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Asymptotic behavior of thinned multi-level point processes in the generalized birthday problem

V. V. Stamatiieva

Igor Sikorsky Kyiv Polytechnic Institute, Kyiv, Ukraine,
ORCID ID: https://orcid.org/0000-0003-2721-8985
(Київський політехнічний інститут імені Ігоря Сікорського, Київ, Україна)

Abstract

The joint asymptotic behavior in the generalized birthday problem is studied using the apparatus of multi-level point processes. The analysis is based on a Poissonized model. We introduce a method that combines a common normalization function with a thinning operation for different completion levels. We prove the vague convergence of the constructed thinned point process to a limiting Poisson process with independent levels. As an application, the joint limiting distribution for the number of classes reaching lower completion levels by a random time is derived.

Keywords: generalized birthday problem, multi-level point process, Poisson process, Poissonization, thinning, vague convergence.

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stamatieva56@gmail.com

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1 Introduction

Combinatorial probability theory, which studies the properties of random variables on finite sets, is a classical branch of modern probability theory. One of the most famous problems in this field is the birthday problem, which in its generalized form is closely related to the coupon collector's problem. In this paper, we consider the following setup: an infinite sequence of objects, where each is independently assigned to one of n classes with a probability of $\frac{1}{n}$. The objects arrive sequentially at discrete time points. For a fixed integer $r \geqslant 1$, we are interested in the first moment in time when some class is represented for the $(r+1)$-th time.

The asymptotic properties of this model have been actively investigated in recent years. Specifically, the case of a single fixed completion level r was analyzed in detail in (Ilienko & Stamatiieva, 2021). A further generalization, presented in (Ilienko & Stamatiieva, 2024), involved studying the joint asymptotic behavior for all levels simultaneously based on r-dependent power normalization.

This article proposes an alternative approach to studying joint asymptotics. We consider a multi-level process where convergence is achieved by using a common normalization for all levels in combination with a thinning operation. This approach proves to be particularly effective for investigating certain specific functionals of the process.

The objective of this work is to prove the convergence of the thinned multi-level point process in a Poissonized model to a limiting Poisson process, and to apply this theoretical result to find the joint limiting distribution for the number of classes that have reached lower completion levels by a random time defined by an event at a fixed higher level.

2 Preliminaries

Let us formally define the key random variables and concepts outlined in the Introduction.

In the discrete-time model of the generalized birthday problem, we denote by $Y_{i,r}^{(n)}$ the arrival time of the $(r+1)$-th object of class i. This variable follows a negative binomial distribution, $Y_{i,r}^{(n)} \sim \mathrm{NegBin}\left(r+1, \frac{1}{n}\right)$. A significant challenge is that the random variables $\left\{Y_{i,r}^{(n)}\right\}_{i=1}^{n}$ are dependent, which complicates direct analysis.

To overcome this dependency, we employ the method of Poissonization, first proposed by L. Holst (Holst, 1986). This technique embeds the problem into a continuous-time framework. In this Poissonized model, the corresponding arrival time $Z_{i,r}^{(n)}$ for each class i has a gamma distribution, $Z_{i,r}^{(n)} \sim \Gamma\left(r+1, \frac{1}{n}\right)$. A crucial property of this framework is that the random variables $Z_{1,r}^{(n)}, \ldots, Z_{n,r}^{(n)}$ are now independent. The coupling between the two models is given by the equation:

$$Z_{i,r}^{(n)} = \sum_{j=1}^{Y_{i,r}^{(n)}} E_j,$$

where $\{E_j\}_{j\geqslant 1}$, is a sequence of independent, standard exponential random variables.

The core of our approach involves the thinning of a point process (see Section 5.3 in (Last & Penrose, 2017)). The thinning of a point process ν with a probability $p \in [0,1]$ is an operation where each point of ν is independently kept with probability p and removed with probability $1-p$. We denote the resulting thinned process as $T_p\nu = p \odot \nu$.

Our investigation of the joint asymptotics relies on constructing a multi-level point process. First, we fix an integer $r_0 \geqslant 1$ and introduce a common normalization function for all levels $r \in \{1, ..., r_0\}$:

$$\psi_{r_0}^{(n)}(x) = \frac{x}{n^{\frac{r_0}{r_0+1}}}, \quad x \in \mathbb{R}. \tag{1}$$

Before we construct the point processes, let us define the mode of convergence used in this paper.

A sequence of locally finite measures $\{\mu_n\}$ on $\mathbb{R}$ is said to converge vaguely to a measure μ (denoted $\mu_n \xrightarrow{v} \mu$) if for every continuous function $f : \mathbb{R} \to [0,\infty)$ with compact support, the following holds:

$$\int_{\mathbb{R}} f(x) d\mu_n(x) \to \int_{\mathbb{R}} f(x) d\mu(x).$$

The convergence of point processes in this paper, denoted by $\xrightarrow{vd}$, is understood as the convergence in distribution with respect to the topology of vague convergence.

Using this, for each level r, we construct a single-level point process $\nu_r^{(n)}$ from the Poissonized variables:

$$\nu_r^{(n)} = \sum_{i=1}^{n} \delta_{\psi_{r_0}^{(n)}(Z_{i,r}^{(n)})},$$

where δ_a stands for the unit mass at a.

Next, we introduce the thinning probability $p_r^{(n)} = n^{-\frac{r_0-r}{r_0+1}}$ for each level r. The main object of our study is the thinned multi-level point process $H_{\text{thin}}^{(n)}$ on the space $\mathbb{X}_{r_0} = \bigcup_{r=1}^{r_0} \{r\} \times \mathbb{R}$. It is defined as the superposition of the individually thinned single-level processes:

$$H_{\text{thin}}^{(n)}\left(\bigcup_{r=1}^{r_0} \{r\} \times B_r\right) = \sum_{r=1}^{r_0} T_{p_r^{(n)}} \nu_r^{(n)}(B_r), \tag{2}$$

where B_r are Borel sets and $T_p(\cdot)$ denotes the thinning operation.

3 Main result

The main theoretical result of this paper establishes the vague convergence of the thinned multi-level point process $H_{\text{thin}}^{(n)}$, defined in (2), to a limiting Poisson process. The theorem is stated as follows.

Theorem 3.1. *Let H be a Poisson point process on the space $\mathbb{X}_{r_0}$ with the intensity measure λ given by*

$$\lambda\left(\bigcup_{r=1}^{r_0}\{r\}\times B_r\right)=\sum_{r=1}^{r_0}\frac{1}{r!}\int_{B_r}x^r\cdot\mathbb{I}\{x\geqslant 0\}dx,\quad B_r\in\mathcal{B}(\mathbb{R}).$$

Then, as $n\to\infty$, the thinned multi-level process converges vaguely in distribution to H:

$$H_{thin}^{(n)}\xrightarrow{vd}H.$$

Remark 3.2. The limiting process H has mutually independent levels, a consequence of the superposition theorem for Poisson processes (see, e.g., Theorem 3.3 in (Last & Penrose, 2017)). Furthermore, each r-th level has a clear interpretation (see Remark 3.2 in (Ilienko, 2019) for a similar interpretation): it is equal in distribution to a standard unit-rate Poisson process on $\mathbb{R}$ after applying the non-linear transformation $h_r(x)=((r+1)!\cdot x)^{\frac{1}{r+1}}$. This follows from the transformation theorem for Poisson processes (see, e.g., Theorem 5.1 in (Last & Penrose, 2017)).

Proof of Theorem 3.1. Our proof is based on a classic criterion for the convergence of point processes. To establish the vague convergence in distribution $H_{\text{thin}}^{(n)}\xrightarrow{vd}H$, it is sufficient to show that the following two conditions hold for any set U from a dissecting ring that generates the Borel σ-algebra on $\mathbb{X}_{r_0}$ (see, e.g., p. 24 in (Kallenberg, 2017)). We will verify them for sets of the form $U=\bigcup_{r=1}^{r_0}(\{r\}\times B_r)$, where each B_r is a finite union of disjoint bounded intervals, which form such a ring:

(i) $\lim_{n\to\infty}\mathbb{P}\{H_{\text{thin}}^{(n)}(U)=0\}=\mathbb{P}\{H(U)=0\}$;

(ii) $\lim_{n\to\infty}\mathbb{E}H_{\text{thin}}^{(n)}(U)=\mathbb{E}H(U)$.

We begin by proving condition (i). The event that the process $H_{\text{thin}}^{(n)}$ has no points in U can be written as:

$$\mathbb{P}\{H_{\text{thin}}^{(n)}(U)=0\}=\mathbb{P}\left\{H_{\text{thin}}^{(n)}\left(\bigcup_{r=1}^{r_0}\{r\}\times B_r\right)=0\right\}=\mathbb{P}\left\{\sum_{r=1}^{r_0}T_{p_r^{(n)}}\nu_r^{(n)}(B_r)=0\right\}$$
$$=\mathbb{P}\{T_{p_r^{(n)}}\nu_r^{(n)}(B_r)=0\quad\forall r=1,...,r_0\}.$$

First, for any $1 \leqslant r_1 < r_2 < \cdots < r_m \leqslant r_0$, let us define the joint probability for a single class i:

$$P^{(n)}_{r_1,\ldots,r_m} = \mathbb{P}\left\{\psi^{(n)}_{r_0}(Z^{(n)}_{i,r_1}) \in B_{r_1}, \ldots, \psi^{(n)}_{r_0}(Z^{(n)}_{i,r_m}) \in B_{r_m}\right\}.$$

This probability does not depend on the index i, since the random variables $Z^{(n)}_{i,r}$ are independent and identically distributed for different i.

Using the independence of the classes and the inclusion-exclusion principle, the void probability is given by:

$$\mathbb{P}\{H^{(n)}_{\text{thin}}(U) = 0\} = \Bigg(1 - \sum_{1\leqslant r_1\leqslant r_0} n^{-\frac{r_0-r_1}{r_0+1}} \cdot P^{(n)}_{r_1} + \sum_{1\leqslant r_1<r_2\leqslant r_0} n^{-\frac{2r_0-r_1-r_2}{r_0+1}} \cdot P^{(n)}_{r_1,r_2} - \ldots$$
$$+(-1)^{r_0} n^{-\frac{r_0^2-r_1-r_2-\cdots-r_0}{r_0+1}} \cdot P^{(n)}_{1,\ldots,r_0}\Bigg)^n.$$

To find the limit of this expression, we analyze its logarithm. Since the sum inside the parenthesis will be shown to vanish as $n \to \infty$, we can use the asymptotic equivalence $\ln(1+\alpha) \sim \alpha, \quad \alpha \to 0$. This yields:

$$\lim_{n\to\infty} \ln \mathbb{P}\{H^{(n)}_{\text{thin}}(U) = 0\}$$
$$= -\sum_{1\leqslant r_1\leqslant r_0} \lim_{n\to\infty} \left(n \cdot n^{-\frac{r_0-r_1}{r_0+1}} \cdot P^{(n)}_{r_1}\right)$$
$$+ \sum_{1\leqslant r_1<r_2\leqslant s} \lim_{n\to\infty} \left(n \cdot n^{-\frac{2r_0-r_1-r_2}{r_0+1}} \cdot P^{(n)}_{r_1,r_2}\right) - \ldots$$
$$+ (-1)^{r_0} \lim_{n\to\infty} \left(n \cdot n^{-\frac{r_0^2-r_1-r_2-\cdots-r_0}{r_0+1}} \cdot P^{(n)}_{1,\ldots,r_0}\right).$$

To prove condition (i), it is therefore sufficient to show that the limit in the first sum equals $\frac{1}{r_1!}\int_{B_{r_1}} x^{r_1} \cdot \mathbb{I}\{x \geqslant 0\}dx$, while the limit in the second sum vanishes. Since all subsequent sums are bounded by the second, this will establish the required statement.

Since the increments of the underlying gamma process, $Z^{(n)}_{i,2} - Z^{(n)}_{i,1}, \ldots, Z^{(n)}_{i,r_0} - Z^{(n)}_{i,r_0-1}$, are independent $\mathsf{Exp}\left(\frac{1}{n}\right)$ random variables, the density $\tilde{f}^{(n)}_{r_1}(x)$ of $\psi^{(n)}_{r_0}(Z^{(n)}_{i,r_1})$ and the joint density $\tilde{f}^{(n)}_{r_1,r_2}(x,y)$ of $(\psi^{(n)}_{r_0}(Z^{(n)}_{i,r_1}), \psi^{(n)}_{r_0}(Z^{(n)}_{i,r_2}))$, respectively, can be derived using a standard change of variables. They are given by:

$$\tilde{f}^{(n)}_{r_1}(x) = \frac{1}{r_1! n^{\frac{r_1+1}{r_0+1}}} \cdot x^{r_1} \cdot \exp\left(-\frac{x}{n^{\frac{1}{r_0+1}}}\right) \mathbb{I}\{x \geqslant 0\},$$

$$\tilde{f}^{(n)}_{r_1,r_2}(x,y) = \frac{n^{-\frac{r_2+1}{r_0+1}}}{r_1!(r_2-r_1-1)!} \cdot x^{r_1} \cdot (y-x)^{r_2-r_1-1} \cdot \exp\left(-\frac{y}{n^{\frac{1}{r_0+1}}}\right) \mathbb{I}\{0 \leqslant x \leqslant y\}.$$

We now use the derived densities to compute the limits required to prove condition (i).

For the terms in the first sum, after substituting the expression for $\tilde{f}_{r_1}^{(n)}(x)$ and canceling the powers of n, we get:

$$\begin{aligned}\lim_{n\to\infty} n \cdot n^{-\frac{r_0-r_1}{r_0+1}} \cdot P_{r_1}^{(n)} &= \lim_{n\to\infty} \frac{1}{r_1!} \int_{B_{r_1}} x^{r_1} \exp\left(-\frac{x}{n^{\frac{1}{r_0+1}}}\right) \mathbb{I}\{x \geqslant 0\}\, dx \\ &= \frac{1}{r_1!} \int_{B_{r_1}} x^{r_1} \mathbb{I}\{x \geqslant 0\}\, dx.\end{aligned}$$

The interchange of the limit and the integral is justified by the dominated convergence theorem, as B_{r_1} is a bounded set.

Next, we show that the terms in the second sum vanish. Using the joint density $\tilde{f}_{r_1,r_2}^{(n)}(x, y)$, we have:

$$\begin{aligned}\lim_{n\to\infty} n \cdot n^{-\frac{2r_0-r_1-r_2}{r_0+1}} \cdot P_{r_1,r_2}^{(n)} = \lim_{n\to\infty} \frac{n^{\frac{r_1-r_0}{r_0+1}}}{r_1!(r_2-r_1-1)!} \iint_{B_{r_1}\times B_{r_2}} x^{r_1}(y-x)^{r_2-r_1-1} \\ \times \exp\left(-\frac{y}{n^{\frac{1}{r_0+1}}}\right) \mathbb{I}\{0 \leqslant x \leqslant y\}\, dx\, dy.\end{aligned}$$

Since $r_1 < r_0$, the exponent $\frac{r_1-r_0}{r_0+1} < 0$ is negative. The integral converges to a finite value as $n \to \infty$ by the dominated convergence theorem. Therefore, the entire expression vanishes. This completes the proof of condition (i).

Finally, we prove condition (ii). By the linearity of expectation and the properties of the thinning operation, we have:

$$\mathbb{E}H_{\text{thin}}^{(n)}(U) = \sum_{r=1}^{r_0} \mathbb{E}\left(T_{p_r^{(n)}} \nu_r^{(n)}\right)(B_r) = \sum_{r=1}^{r_0} p_r^{(n)} \mathbb{E}\nu_r^{(n)}(B_r).$$

Since the process $\nu_r^{(n)}(B_r)$ is a sum of n independent and identically distributed indicator variables, it follows a binomial distribution with expectation $\mathbb{E}\nu_r^{(n)}(B_r) = n \cdot P_r^{(n)}$. Therefore,

$$\lim_{n\to\infty} \mathbb{E}H_{\text{thin}}^{(n)}(U) = \lim_{n\to\infty} \sum_{r=1}^{r_0} n \cdot n^{-\frac{r_0-r}{r_0+1}} \cdot P_r^{(n)} = \sum_{r=1}^{r_0} \frac{1}{r!} \int_{B_r} x^r \mathbb{I}\{x \geqslant 0\} dx = \mathbb{E}H(U).$$

The limit of each term in the sum was established during the proof of condition (i). Since both conditions of the convergence criterion are satisfied, the proof of Theorem 3.1 is complete.

□

4 Application of the main result

As an application of the convergence established in Theorem 3.1, we can now derive the joint limiting distribution for a key functional of the process. We will investigate the number of classes that have reached completion level $r+1$ by a random time, which is defined as the moment the m-th class completes a higher level, r_0+1.

Formally, for a fixed level $r_0 \geqslant 1$ and a rank $m \geqslant 1$, let $Z^{(n)}_{(m),r_0}$ denote the m-th order statistic among the completion times $\{Z^{(n)}_{i,r_0}\}_{i=1}^n$. We are interested in the asymptotic behavior of the random vector of counts:

$$Q^{(n)}_{r,m} = \operatorname{card}\{i : Z^{(n)}_{i,r} < Z^{(n)}_{(m),r_0}\}, \quad r = 1, \dots, r_0 - 1.$$

The following theorem describes its limiting distribution.

Theorem 4.1. *Let G_m be a random variable with a Gamma distribution, $G_m \sim \Gamma(m, 1)$. Then, as $n \to \infty$, the following convergence in distribution holds in the space $\mathbb{R}^{r_0-1}$:*

$$\left(n^{-\frac{r_0-r}{r_0+1}} \cdot Q^{(n)}_{r,m}, \quad r = 1, \dots, r_0 - 1\right) \xrightarrow{d} \left(\frac{((r_0+1)! \cdot G_m)^{\frac{r+1}{r_0+1}}}{(r+1)!}, \quad r = 1, \dots, r_0 - 1\right).$$

Proof. The proof relies on applying the Continuous Mapping Theorem to the process convergence from Theorem 3.1 and then using a result that connects the convergence of thinned random variables to their normalized counterparts.

Let us first define two functionals of the process $H^{(n)}_{\text{thin}}$. Let $\tilde{T}^{(n)}_{r_0,m}$ be the m-th point of the process on the fixed level r_0. Since the thinning probability at this level is $p^{(n)}_{r_0} = n^{-\frac{r_0-r_0}{r_0+1}} = 1$, this point corresponds to the normalized m-th order statistic, $\tilde{T}^{(n)}_{r_0,m} = \psi^{(n)}_{r_0}(Z^{(n)}_{(m),r_0})$.

Next, consider the vector with components $\tilde{V}^{(n)}_{r,m} = H^{(n)}_{\text{thin}}\left(\{r\} \times (0, \tilde{T}^{(n)}_{r_0,m})\right)$, $r = 1, \dots, r_0 - 1$. By construction, $\tilde{V}^{(n)}_{r,m}$ is the number of points of the thinned process $T_{p^{(n)}_r}\nu^{(n)}_r$ on the random interval $(0, \tilde{T}^{(n)}_{r_0,m})$. This means that $\tilde{V}^{(n)}_{r,m}$ has the same distribution as the thinned version of the original count, $p^{(n)}_r \odot Q^{(n)}_{r,m}$.

By Theorem 3.1 and the Continuous Mapping Theorem (see, e.g., Theorem 4.27 in (Kallenberg, 2017)), the vector of functionals $(\tilde{V}^{(n)}_{r,m})$ converges in distribution to a limiting vector $(\tilde{V}_{r,m})$, where $\tilde{V}_{r,m} = H\left(\{r\} \times (0, \tilde{T}_{r_0,m})\right)$ and $\tilde{T}_{r_0,m}$ is the m-th point of the limit process H on level r_0.

The structure of the limiting process H (see Remark 3.2) determines the distribution of this vector. The random time $\tilde{T}_{r_0,m}$ has the distribution of a transformed

Gamma variable, $\tilde{T}_{r_0,m} \overset{d}{=} h_{r_0}(G_m) = ((r_0+1)! \cdot G_m)^{\frac{1}{r_0+1}}$, where $G_m \sim \Gamma(m,1)$. Since the levels of H are independent, the limiting counts $\tilde{V}_{r,m}$ are conditionally independent random variables that follow a Poisson distribution with parameter $\lambda_r\left((0,\tilde{T}_{r_0,m})\right) = \frac{(\tilde{T}_{r_0,m})^{r+1}}{(r+1)!}$.

Thus, we have established the convergence for the vector of thinned counts:

$$(p_1^{(n)} \odot Q_{1,m}^{(n)}, \ldots, p_{r_0-1}^{(n)} \odot Q_{r_0-1,m}^{(n)}) \xrightarrow{d} (\tilde{V}_{1,m}, \ldots, \tilde{V}_{r_0-1,m}),$$

where the components of the limiting vector have a mixed Poisson distribution, which can be represented as

$$\tilde{V}_{r,m} \sim N_r\left(\frac{((r_0+1)! \cdot G_m)^{\frac{r+1}{r_0+1}}}{(r+1)!}\right),$$

with N_r being independent unit-rate Poisson processes.

With the convergence of the thinned counts established, the final step is to invoke the result that connects this to the convergence of their normalized counterparts (see, e.g., Theorem 4.1 in (Ilienko, 2020)). Since its technical conditions are satisfied, the theorem's application completes the proof of Theorem 4.1. □

5 Summary

This paper presents an effective method for analyzing the joint asymptotics in the generalized birthday problem by constructing a thinned multi-level point process. The established convergence of this process allows for the derivation of the joint limiting distribution for key process functionals, such as the number of completion events by a random time. A natural direction for further research is the extension of these results to the original, non-poissonized model.

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В. В. Стаматієва (2025). Асимптотична поведінка проріджених багаторівневих точкових процесів в узагальненій задачі про дні народження. *Mathematics in Modern Technical University, 2025*(2), 55–63.

Анотація. У статті досліджено спільну асимптотичну поведінку характеристик узагальненої задачі про дні народження. Аналіз проведено в рамках пуассонізованої моделі з використанням апарату багаторівневих точкових процесів, що дозволяє уникнути проблеми залежності, властивої вихідній постановці задачі. Запропоновано підхід до вивчення спільних асимптотик, який полягає в застосуванні спільної нормуючої функції у поєднанні з операцією проріджения, ймовірність якої залежить від досліджуваного рівня заповнення.

Основним теоретичним результатом є доведення грубої збіжності за розподілом побудованого таким чином точкового процесу до граничного пуассонівського процесу, структурною особливістю якого є незалежність його рівнів. Як застосування цієї теореми отримано спільний граничний розподіл для кількості типів, що досягли нижчих рівнів заповнення до випадкового моменту, визначеного m-тим заповненням на вищому рівні. Показано, що граничний розподіл є змішаним пуассонівським, де змішуючою виступає гамма-розподілена випадкова величина.

Ключові слова: узагальнена задача про дні народження, багаторівневий точковий процес, пуассонівський процес, пуассонізація, проріджения, груба збіжність.