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On the convergence of stochastic Fibonacci series

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Abstract

One very spesific case of linear second-order autoregressive sequence of random variables, the so-called stochastic Fibonacci sequence, is considered. For a random series, whose terms are properly normed partial sums of elements of the Fibonacci sequence, we study assumptions which garantee almost sure convergence of the introduced series. Obtained result is quite simple and understandable. In partcular, it implies that the Strong Law of Large Numbers for the Fibonacci sequence holds true.

Keywords: linear autoregression models; Strong Law of Large Numbers; sums of independent random variables, almost sure convergence of random series.

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1 Introduction

On a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider one particular random sequence $(\xi_k) = (\xi_k, k \geqslant 1)$, described by the following system of recurrence equations

$$\xi_0 = 0, \quad \xi_1 = \theta_1, \quad \xi_k = \xi_{k-1} + \xi_{k-2} + \theta_k, \quad k \geqslant 2, \tag{1}$$

where (θ_k) is a sequence of independent symmetric random variables, $\mathbb{P}\{\theta_k = 0\} < 1$, $k \geqslant 1$. Implying some analogy, let us call (ξ_k) the stochastic Fibonacci sequence. Obviously, (ξ_k) belongs to the class of linear autoregression models, which are widely studied throughout the literature, see, for instance, (Buraczewski, Damek, & Mikosch, 2016), and references therein. Set

$$S_n = \sum_{k=1}^{n} \xi_k, \quad n \geqslant 1,$$

In this paper we study almost sure (a.s.) convergence of the series

$$\sum_{n=1}^{\infty} \frac{S_n}{n^{1+\varepsilon}\varphi^n}, \tag{2}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ration, and $\varepsilon > 0$.

Series of similar kind for sums (S_n) whose terms are independent random variables or random elements in Banach spaces appear, for instance, in (Li, Qi, & Rosalsky, 2011). Namely, authors obtained general assumptions under which the series $\sum_{n=1}^{\infty} \frac{\|S_n\|}{n^{1+\varepsilon}}$, converges a.s. In (Ilienko, 2017) the a.s. convergence of the series $\sum_{n=1}^{\infty} \frac{S_n}{n^{1+\varepsilon}}$ was investigated for sums (S_n), whose terms are elements of linear first-order autoregression sequences of random variables.

In this paper we focus only on stochastic Fibonacci sequence, although applied methods allow us to consider the most general case at once, say, any m-th order autoregressive sequence.

2 Preliminaries

In what follows, we will use 2 auxilliary facts, which are presented in this section.

First of all we recall the well known Kahane's contraction principle (Kahane, 1985; Buldygin & Solntsev, 1997). Let $\mathfrak{X}$ be a separable Banach space endowed with the norm $\|\cdot\|$, and (X_k) be a sequence of independent symmetric $\mathfrak{X}$-valued random elements, $S_n = X_1 + X_2 + ... + X_n$, $n \geqslant 1$. Let also (c_k) be a non-random sequence such that $\sup\limits_{k\geqslant 1}|c_k| < \infty$, and $\widetilde{S}_n = c_1 X_1 + c_2 X_2 + ... + c_n X_n$, $n \geqslant 1$.

Proposition 2.1. *[Kahane's contraction principle]*

If the sequence (S_n) converges a.s. in the norm of the space $\mathfrak{X}$, then the sequence $(\widetilde{S}_n)$ converges a.s. in the norm of the space $\mathfrak{X}$.

Secondly, let us present a criterion for the convergence a.s. of the series, whose terms are elements of linear autoregression sequence of random variables. The result below is adapted to the case of the sequence of order 3, the general case is considered in (Buldygin & Runovska, 2014; Ilienko, 2016).

Thus, consider a zero-mean third order autoregression sequence of random variables (ζ_k), i.e. the sequence which obeys the system of following recurrence equations:

$$\zeta_{-2} = \zeta_{-1} = \zeta_0 = 0, \quad \zeta_k = b_k^{(1)}\zeta_{k-1} + b_k^{(2)}\zeta_{k-2} + b_k^{(3)}\zeta_{k-3} + \beta_k\theta_k, \quad k \geqslant 1,$$

where (β_k) is a nonrandom real sequence, $(b_k^{(j)};\ \ j = 1, 2, 3,\ \ k \geqslant 1)$ is a nonrandom real-valued set, and (θ_k) is a sequence of independent symmetric random variables such that $\mathbb{P}\{\theta_k = 0\} < 1$, $k \geqslant 1$. Let $\mathfrak{R}^\infty$ be the class of all sequences of positive integers increasing to infinity.

Proposition 2.2. *The series $\sum_{k=1}^{\infty} \zeta_k$ converges a.s. if and only if the following three assumptions hold:*

1) *for any $k \geqslant 1$ the nonrandom series $\left(\sum_{l=0}^{\infty} \beta_k u_{k+l}^{(k+1)}\right)$ is convergent, where $(u_n^{(k+1)}, n \geqslant k-1)$ is a nonrandom sequence which obeys the system of recurrence equations*

$$u_n^{(k+1)} = b_n^{(1)} u_{n-1}^{(k+1)} + b_n^{(2)} u_{n-2}^{(k+1)} + b_n^{(3)} u_{n-3}^{(k+1)}, \quad n \geqslant k+1,$$
$$u_{k-2}^{(k+1)} = u_{k-1}^{(k+1)} = 0, \quad u_k^{(k+1)} = 1;$$

2) *the series $\sum_{k=1}^{\infty} U_k\theta_k$ converges a.s., with $U_k = \sum_{l=0}^{\infty} \beta_k u_{k+l}^{(k+1)}$, $k \geqslant 1$;*

3) *for all the sequences (m_j) from the class $\mathfrak{R}^\infty$*

$$\lim_{j\to\infty} \Bigg| \sum_{k=m_j+1}^{m_{j+1}} \Bigg(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \Bigg) \beta_k\theta_k \Bigg| = 0, \quad a.s.$$

3 Main result

Set

$$\hat{U}_k = \sum_{j=\left\lfloor \frac{k+1}{2} \right\rfloor}^{\infty} \frac{1}{j^{1+\varepsilon}}, \quad k \geqslant 1,$$

where $\lfloor \cdot \rfloor$ stands for the floor function.

Let us directly proceed to the main result.

Theorem 3.1. *If in* (1) $\mathbb{E}\theta_n^2 < \infty$, $n \geqslant 1$, *and*

$$\sum_{n=1}^{\infty} \hat{U}_n^2 \, \mathbb{E}\theta_n^2 < \infty,$$

then the series (2) *converges a.s.*

Proof. First note that the sequence of partial sums (S_n) of the Fibonacci sequence obeys the system of recurrence equations of the 3-d order:

$$S_{-1} = S_0 = 0, \quad S_1 = \theta_1, \quad S_n = 2S_{n-1} - S_{n-3} + \theta_n, \quad n \geqslant 2.$$

If we denote

$$c_n = \frac{1}{n^{1+\varepsilon}\varphi^n}, \quad n \geqslant 1,$$

and $\tilde{S}_n = c_n S_n$, then the sequence $(\tilde{S}_n)$ in its turn obeys the system of the following recurrence equations

$$\tilde{S}_n = \Big(\frac{2c_n}{c_{n-1}}\Big)\tilde{S}_{n-1} + \Big(\frac{-c_n}{c_{n-3}}\Big)\tilde{S}_{n-3} + c_n\theta_n, \quad n \geqslant 2.$$

Thus, the series (2) converges a.s. if and only if the series $\sum_{n=1}^{\infty} \tilde{S}_n$ converges a.s. According to Proposition 2.2 the series $\sum_{n=1}^{\infty} \tilde{S}_n$ converges a.s. if and only if the assumptions 1)-3) hold true, where

$$b_n^{(1)} = \frac{2c_n}{c_{n-1}}, \quad b_n^{(2)} = 0, \quad b_n^{(3)} = \frac{-c_n}{c_{n-3}}, \quad \beta_n = c_n, \quad n \geqslant 1.$$

Since,

$$u_{k+l}^{(k+1)} = \big(F_{k+l-3} - 1\big)\frac{c_{k+l}}{c_k}, \quad l \geqslant 0,$$

then the series $\Big(\sum_{l=0}^{\infty} \beta_k u_{k+l}^{(k+1)}\Big)$ is of the following form

$$\sum_{l=0}^{\infty} \Big(\frac{F_{k+l-3} - 1}{\varphi^{k+l}}\Big)\frac{1}{(k+l)^{1+\varepsilon}}, \tag{3}$$

where F_m is the m-th Fibonacci number, starting with $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and so on.

Ommiting straightforward calculations, one has that

$$\frac{F_{k+l-3} - 1}{\varphi^{k+l}} = \frac{1 + (-1)^{k+l}}{\varphi^3\sqrt{5}}, \quad l \geqslant 0.$$

Therefore, the series (3) converges for any $k \geqslant 1$, and $\hat{U}_k$'s are well-defined. Thus, assumption 1) of Proposition 2.2 holds true. Now, U_k's from Proposition 2.2 are of the following form

$$U_k = \frac{1}{\varphi^3\sqrt{5}} \sum_{m=k}^{\infty} \frac{1+(-1)^m}{m^{1+\varepsilon}} = \frac{1}{2^{\varepsilon}(5+2\sqrt{5})} \sum_{j=\left\lfloor \frac{k+1}{2} \right\rfloor}^{\infty} \frac{1}{j^{1+\varepsilon}} = \frac{\hat{U}_k}{2^{\varepsilon}(5+2\sqrt{5})},\ k \geqslant 1,$$

and due to the assumptions of the Theorem and the well-known two-series Theorem, the series $\sum_{k=1}^{\infty} \hat{U}_k\theta_k$ converges a.s. Therefore the assumption 2) of Proposition 2.2 holds true as well.

Finally, fix arbitrary sequence (m_j) from the class $\mathfrak{R}^{\infty}$. Since for any $k \geqslant 1$,

$$\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)}\beta_k = \sum_{l=0}^{m_{j+1}-k} (F_{k+l-3}-1)c_{k+l} \leqslant \sum_{l=0}^{\infty} \left(\frac{F_{k+l-3}-1}{\varphi^{k+l}}\right)\frac{1}{(k+l)^{1+\varepsilon}} = U_k,$$

according to Kahane's contraction principle, assumption 3) of Proposition 2.2 follows immediately from the convergence of the series $\sum_{k=1}^{\infty} \hat{U}_k\theta_k$.

The proof is complete.

□

In conclusion, let us emphasize that the series, whose terms are elements of the Fibonacci sequence itself, i.e. the series $\sum_{n=1}^{\infty} \xi_n$, is divergent a.s., as well as the series $\sum_{n=1}^{\infty} \frac{S_n}{n^{1+\varepsilon}}$ (not normalized by φ^n in comparison to (2)). It is clear even intuitively, since with increase of n the coefficients of ξ_n behave as Fibonacci terms itself.

Observe also, that the sequence $(\hat{U}_n,\ n \geqslant 1)$ is monotonically decreasing, which yields that the sufficient conditions of Theorem 3.1 can be further simplified. Indeed, for the series (2) to converge a.s. it is sufficient that the series $\sum_{k=1}^{\infty} \mathbb{E}\theta_k^2$ converges, or equivalently, that the sequence $\left(\sum_{k=1}^{n} \mathbb{E}\theta_k^2,\ n \geqslant 1\right)$ is uniformly bounded. But, if θ_k, $k \geqslant 1$, are identically distributed, then series (2) converges a.s. if $\sum_{n=1}^{\infty} \hat{U}_n^2 < \infty$, which in its turn is possible only for $\varepsilon > 1/2$.

4 Summary

Although the very specific case of the autoregressive sequence is considered, the result is quite pleasant and simple. In particular, it implies that if the assumptions of Theorem 3.1 are satisfied, then $\frac{S_n}{n^{1+\varepsilon}\varphi^n} \underset{n\to\infty}{\longrightarrow} 0$ a.s., that is the Strong Law of Large Numbers holds true.

Further investigation of sequence (1) and other linear regression sequences of similar type might deal with complete convergence, as considered in (Ilienko, 2021) for linear first order autoregression sequences.

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М. К. Ільєнко, Л. А. Руновська (2025). Про збіжність стохастичного ряду Фібоначі. *Mathematics in Modern Technical University, 2025*(1), 43–49.

Анотація. У статті розглянуто один частковий випадок лінійної авторегресійної послідовності випадкових величин другого порядку, так звана стохастична послідовність Фібоначчі. А саме, нехай послідовність випадкових величин $(\xi_k) = (\xi_k, k \geqslant 1)$ задовольняє систему рекурентних співвідношень:

$$\xi_0 = 0, \quad \xi_1 = \theta_1, \quad \xi_k = \xi_{k-1} + \xi_{k-2} + \theta_k, \quad k \geqslant 2,$$

де (θ_k) — послідовність незалежних симетрично розподілених випадкових величин. Нехай $S_n = \sum_{k=1}^n \xi_k$, $n \geqslant 1$. Розглянемо випадковий ряд

$$\sum_{n=1}^{\infty} \frac{S_n}{n^{1+\varepsilon}\varphi^n},$$

де $\varphi = \frac{1+\sqrt{5}}{2}$ — золотий переріз, та $\varepsilon > 0$. У статті вивчаються умови, за яких цей ряд є збіжним майже напевно. А саме, встановлено, що для збіжності ряду майже напевно достатньо, щоб виконувались наступні умови:

$$\mathbb{E}\theta_k^2 < \infty,\ k \geqslant 1, \qquad \text{та} \qquad \sum_{n=1}^{\infty} \hat{U}_n^2\, \mathbb{E}\theta_n^2 < \infty,$$

де $\hat{U}_k = \sum_{j=\left\lfloor \frac{k+1}{2} \right\rfloor}^{\infty} \dfrac{1}{j^{1+\varepsilon}}$, $k \geqslant 1$. Зокрема зі збіжності розгянутого ряду одразу випливає посилений закон великих чисел для стохастичної послідовності Фібоначчі.

Ключові слова: лінійні авторегресійні моделі; посилений закон великих чисел; суми випадкових величин; збіжність майже напевно випадкових рядів.