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Dirichlet problem in a ball for Laplace's equation with Laplacian with respect to a measure

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Abstract

In this article, we study a Dirichlet problem for a generalized Laplace's equation. We consider a construction of Laplacian with respect to a measure, that generalizes the classical Laplace's operator to the case of an arbitrary measure. Certain properties of the constructed Laplacian are studied and a Dirichlet problem for Laplace's equation with this new Laplacian is set.

We propose a general solution construction framework for the Dirichlet problem in a ball in 2- and 3-dimensional spaces in the case of densities, that are invariant to orthogonal transforms. Using this framework we find explicit solutions for several important and rich families of densities, with the Gaussian density among them.

Keywords: measure; divergence; Laplacian; Laplace's equation; Dirichlet problem.

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1 Introduction

The construction of divergence with respect to (w.r.t.) a measure and Laplacian w.r.t. a measure allows us to generalize classical divergence and Laplacian operators to the case of non-invariant measure. The problem of generalizing classical results of mathematical physics to the case of non-invariant measure is quite promising from the standpoint of the possibility of transferring certain results to the case of an infinite-dimensional argument. For more detail, see the following papers: (Bogdanskii, 2012), (Bogdanskii, 2013), (Bogdanskii & Sanzharevskii, 2014), (Bogdanskii & Potapenko, 2016) and (Bogdanskii & Potapenko, 2017).

Let us now describe the construction of divergence w.r.t. a measure and Laplacian w.r.t. a measure we will be using in this paper.

Let $(X, \mathfrak{A}, \mu)$ be a space with a measure, where $X = \mathbb{R}^m$, $\mathfrak{A}$ is a Borel σ-algebra of subsets of X, and μ is a signed measure (from now on — just «measure») on $\mathfrak{A}$ (whether finite or infinite).

Let us also consider $Z \in C_b^1(X, X)$, where $C_b^1(X, X)$ denotes a space of all vector fields on X with values in X that are continuously differentiable, and are bounded on X together with their first derivatives. By $\Phi_t^Z(x_0)$ we denote the flow of the vector field Z at the time t valued at the point $x_0 \in X$.

Let's consider the next initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t) = Z(x(t)) \quad \forall t \in \mathbb{R} \\ x(0) = x_0 \end{cases}$$

Since $Z \in C_b^1(X, X)$, there exists a unique solution of this problem $x(t) = \Phi(t, x_0) = \Phi_t^Z(x_0)$.

Thus, for every fixed $t \in \mathbb{R}$ we've got a map $\mathbb{R}^m \ni x_0 \mapsto \Phi_t^Z(x_0) \in \mathbb{R}^m$. One can prove that it is a diffeomorphism.

Considering t as a parameter, we obtain a one-parameter family of diffeomorphisms Φ_t^Z. This family is called a «flow of vector field Z».

Since for every $t \in \mathbb{R}$: Φ_t^Z is a diffeomorphism, one can easily prove next

Proposition 1.1. *If $A \in \mathfrak{A}$, then for each $t \in \mathbb{R}$: $\Phi_t^Z(A) \in \mathfrak{A}$. Furthermore, for every $t \in \mathbb{R}$ a map $\mathfrak{A} \ni A \mapsto \mu(\Phi_t^Z(A))$ is a measure.*

We will also need the next theorem.

Theorem 1.2 (Nikodym, Vitali)**.** *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on $\mathfrak{A}$ and for every $A \in \mathfrak{A}$ there exists $\lim\limits_{n\to\infty} \mu_n(A) =: \mu(A)$. Then μ is a measure.*

Proof. See (Bogachev, 2007). □

Definition 1.3. Measure μ is said to be «differentiable» along a field Z if for every $A \in \mathfrak{A}$ there exists $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \mu(\Phi_t^Z(A)) =: \vartheta(A)$. Measure ϑ in such a case is called a «derivative» of measure μ along a field Z.

Remark 1.4. The fact that ϑ is a measure immediatly follows from Nikodym–Vitali theorem.

It turns out that the derivative ϑ is absolutely continuous w.r.t. μ (see for example (Bogachev, 2010)). Its density $\frac{\mathrm{d}\vartheta}{\mathrm{d}\mu}$ is denoted as $\operatorname{div}_\mu Z$ (read «divergence of Z w.r.t. μ»).

Let's now consider a measure that has a density f w.r.t. some measure μ, which is differentiable along a field Z (we denote such a measure by $f \cdot \mu$). It turns out that in such a case we can rewrite $\operatorname{div}_{f\cdot\mu} Z$ in terms of $\operatorname{div}_\mu Z$.

Proposition 1.5. *Let $f : X \to \mathbb{R}$; $f \in C_b^1(X)$, where $C_b^1(X)$ denotes a space of all continuously differentiable real-valued functions on X that are bounded on X together with their first derivative. If measure μ is differentiable along Z then measure $f \cdot \mu$ is differentiable along Z and the next equality holds*

$$\operatorname{div}_{f\cdot\mu} Z = \operatorname{div}_\mu(fZ) = f \operatorname{div}_\mu Z + (\overrightarrow{\operatorname{grad}} f, Z) \tag{1}$$

Proof. See (Bogdanskii & Sanzharevskii, 2014). □

It is also known (and one can easily prove it) that if a field $Z \in C_b^1(X)$ then the Lebesgue measure λ («volume») is differentiable along Z and, furthermore, $\operatorname{div}_\lambda Z = \operatorname{div} Z = \sum\limits_{k=1}^{m} \frac{\partial Z_k}{\partial x_k}$.

Definition 1.6. Laplace's operator w.r.t. a measure μ is defined as follows

$$\Delta_\mu : C^2(X) \to C(X);\ \Delta_\mu := \operatorname{div}_\mu \circ \overrightarrow{\operatorname{grad}}.$$

Remark 1.7. From the definition 1.6 we see that Laplace's operator w.r.t. a measure μ is well-defined only for those functions $u \in C^2(X)$ for which the field $\overrightarrow{\operatorname{grad}}\, u$ is bounded on X and μ is differentiable along $\overrightarrow{\operatorname{grad}}\, u$.

Let B_R be an open ball with the radius R and the center at $\vec{0} \in \mathbb{R}^m$. Let λ be the Lebesgue measure on $\mathbb{R}^m$.

Let us now consider the measure $\mu = f \cdot \lambda$, where $f \in C^1(\overline{B_R})$ and f is invariant with respect to orthogonal transforms, i.e. $f(\vec{x}) = g(\|\vec{x}\|)$.

We now consider the next Dirichlet problem. Find a function $u : \overline{B_R} \to \mathbb{R}$; $u \in C^2(B_R) \cap C(\overline{B_R})$ such that

$$\begin{cases} \Delta_\mu u(\vec{x}) = \operatorname{div}_\mu(\overrightarrow{\operatorname{grad}}\, u(\vec{x})) = 0\ \forall \vec{x} \in B_R, \\ u\big|_{\partial B_R} = h \end{cases} \tag{2}$$

where h is some predefined continuous function on border ∂B_R of the ball.

The uniqueness of the solution of the problem (2) immediately follows from the maximum principle for Laplacian w.r.t. a measure (see (Bogdanskii, 2016)).

According to proposition 1.5 we can rewrite problem (2) in the next form

$$\begin{cases} f\Delta u + (\overrightarrow{\operatorname{grad}} f, \overrightarrow{\operatorname{grad}} u) = 0 \ \forall \vec{x} \in B_R, \\ u\big|_{\partial B_R} = h \end{cases} \tag{3}$$

2 Solutions Construction Framework

For 2- and 3-dimentional cases there were obtained a general framework for constructing solutions of the Dirichlet problem (3). With its help all the complexity of solving a Dirichlet problem in these cases can be reduced to solving an ordinary differential equation of the special type. So, let's describe these framework (or «recipe» for constructing solutions) in more detail.

2.1 2-dimensional case

First, we have to solve next ordinary differential equation

$$\rho^2 P''(\rho) + \rho(1 + \rho(\ln f(\rho))')P'(\rho) - n^2 P(\rho) = 0,\ n \in \mathbb{N} \cup \{0\} \tag{4}$$

More precisely, for each $n \in \mathbb{N} \cup \{0\}$ we have to find a solution $P_n(\rho)$ of equation (4), that is bounded on the segment $[0, R]$ together with its derivative and such that $P_n(R) \neq 0$.

Lemma 2.1. *Equation (4) cannot have two linearly independent solutions, that are bounded on* $[0, R]$ *together with their first derivative.*

Once we have found such a solution $P_n(\rho)$, then the solution of the Dirichlet problem (3) is defined by the next formula

$$u(\rho, \varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \frac{P_n(\rho)}{P_n(R)} (\alpha_n \cos n\varphi + \beta_n \sin n\varphi),$$

where

$$\alpha_n = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \cos n\varphi \, \mathrm{d}\varphi,\ n \geqslant 0,$$

$$\beta_n = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \sin n\varphi \, \mathrm{d}\varphi,\ n \geqslant 1.$$

2.2 3-dimensional case

Similarly to 2-dimensional case, first we have to solve ordinary differential equation of the next form

$$r^2 P''(r) + r(2 + r(\ln f(r))')P'(r) - n(n+1)P(r) = 0,\ n \in \mathbb{N} \cup \{0\} \tag{5}$$

Analogically, for each $n \in \mathbb{N} \cup \{0\}$ we have to find a solution $P_n(r)$ of equation (5), that is bounded on the segment $[0, R]$ together with its derivative and such that $P_n(R) \neq 0$.

Similarly, we have the next result.

Lemma 2.2. *Equation (5) cannot have two linearly independent solutions, that are bounded on* $[0, R]$ *together with their first derivative.*

Once we have found such a solution $P_n(r)$, then the solution of the Dirichlet problem (3) is defined by the nex formula

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} a_{nk} \frac{P_n(r)}{P_n(R)} Y_{nk}(\theta, \varphi),$$

where

$$a_{nk} = \frac{\displaystyle\iint\limits_{[0,\pi]\times[0,2\pi]} f(\theta, \varphi) Y_{nk}(\theta, \varphi)\, \mathrm{d}\theta\, \mathrm{d}\varphi}{\displaystyle\iint\limits_{[0,\pi]\times[0,2\pi]} (Y_{nk}(\theta, \varphi))^2\, \mathrm{d}\theta\, \mathrm{d}\varphi},\quad n \in \mathbb{N} \cup \{0\},\ |k| \leqslant n.$$

Here $Y_{nk}(\theta, \varphi)$ denotes spherical harmonics. For more detail on spherical harmonics see (Sveshnikov, Bogolyubov, & Kratsov, 2004).

3 Explicit Solutions

Using the framework, described above, there were obtained explicit solutions of the Dirichlet problem in a ball for some important special cases of densities f.

Theorem 3.1. *Let* $m = 2,\ f(\vec{x}) = A\|\vec{x}\|^B$, *where* $A > 0,\ B \geqslant 0$. *Then solution of the problem (3) in polar coordinates is as follows*

$$u(\rho, \varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{R}\right)^{\lambda_n} (\alpha_n \cos n\varphi + \beta_n \sin n\varphi),$$

where

$$\lambda_n = \frac{-B + \sqrt{B^2 + 4n^2}}{2},\ n \geqslant 1,$$

$$\alpha_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(\varphi)\cos n\varphi \,\mathrm{d}\varphi,\ \ n \geqslant 0,$$

$$\beta_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(\varphi)\sin n\varphi \,\mathrm{d}\varphi,\ \ n \geqslant 1.$$

Theorem 3.2. *Let $m = 2$, $f(\vec{x}) = Ae^{-B\|\vec{x}\|^d}$, where $A > 0$, $B > 0$, $d > 0$. Then solution of the problem (3) in polar coordinates is as follows*

$$u(\rho,\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^n \frac{{}_1F_1(\frac{n}{d}, \frac{2n+d}{d}, B\rho^d)}{{}_1F_1(\frac{n}{d}, \frac{2n+d}{d}, BR^d)}(\alpha_n\cos n\varphi + \beta_n \sin n\varphi)$$

(here ${}_1F_1$ denotes confluent hyperheometric function of the first kind),
where

$$\alpha_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(\varphi)\cos n\varphi \,\mathrm{d}\varphi,\ \ n \geqslant 0,$$

$$\beta_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(\varphi)\sin n\varphi \,\mathrm{d}\varphi,\ \ n \geqslant 1.$$

Theorem 3.3. *Let $m = 3$, $f(\vec{x}) = A\|\vec{x}\|^B$, where $A > 0$, $B \geqslant 0$. Then solution of the problem (3) in spherical coordinates is as follows*

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{k=-n}^{n} a_{nk}\left(\frac{r}{R}\right)^{\lambda_n} Y_{nk}(\theta,\varphi),$$

where

$$\lambda_n = \frac{-1 - B + \sqrt{(1+B)^2 + 4n(n+1)}}{2},\ \ n \geqslant 0,$$

$$a_{nk} = \frac{\iint\limits_{[0,\pi]\times[0,2\pi]} f(\theta,\varphi)Y_{nk}(\theta,\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi}{\iint\limits_{[0,\pi]\times[0,2\pi]} (Y_{nk}(\theta,\varphi))^2\,\mathrm{d}\theta\,\mathrm{d}\varphi},\quad n \in \mathbb{N}\cup\{0\},\ \ |k| \leqslant n.$$

Theorem 3.4. *Let $m = 3$, $f(\vec{x}) = Ae^{-B\|\vec{x}\|^d}$, where $A > 0$, $B > 0$, $d > 0$. Then solution of the problem (3) in spherical coordinates is as follows*

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{k=-n}^{n} a_{nk}\left(\frac{r}{R}\right)^n \frac{{}_1F_1(\frac{n}{d}, \frac{n+d+1}{d}, Br^d)}{{}_1F_1(\frac{n}{d}, \frac{n+d+1}{d}, BR^d)} Y_{nk}(\theta,\varphi),$$

where

$$a_{nk} = \frac{\displaystyle\iint\limits_{[0,\pi]\times[0,2\pi]} f(\theta,\varphi) Y_{nk}(\theta,\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi}{\displaystyle\iint\limits_{[0,\pi]\times[0,2\pi]} (Y_{nk}(\theta,\varphi))^2\,\mathrm{d}\theta\,\mathrm{d}\varphi}, \quad n \in \mathbb{N}\cup\{0\},\ |k| \leqslant n.$$

4 Summary

In this article the Dirichlet problem for Laplace's equation with Laplacian of a special form was studied. Certain important properties of the new Laplacian were presented, which helped to rewrite our Dirichlet problem in the form of the classical mathematical physics problem.

We presented the general framework (or scheme) for constructing solutions for the Dirichlet problem in a ball in 2- and 3-dimensional spaces in the case of densities, which are invariant to orthogonal transforms. Then using this framework we obtained explicit solutions for this Dirichlet problem for several important and rich families of densities, one of which includes, among others, the Gaussian density.

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В. Шрам (2018). Задача Діріхле в кулі для рівняння Лапласа з лапласіаном за мірою. *Mathematics in Modern Technical University*, 2018(1), 11–18.

Анотація. Однією з найбільш відомих задач математичної фізики є задача Діріхле для рівняння Лапласа. Рівняння Лапласа описує безліч стаціонарних фізичних процесів і виникає в багатьох задачах механіки, теплопровідності, електростатики, гідравліки тощо. Незважаючи на те, що цю задачу вважають класичною, під час її розв'язанні виникає багато труднощів, якщо досліджувана область має форму, складнішу, аніж круг, куля, кільце, прямокутник тощо.

У роботі розглядається узагальнення задачі Діріхле для рівняння Лапласа. Для цого використано конструкцію лапласіана за мірою, яка узагальнює звичайний оператор Лапласа на випадок довільної міри. З фізичної точки зору це дає можливість розглядати задачу в областях, які не є однорідними — мають змінну теплопровідність, електропровідність тощо.

Далі сформульовано постановку задачі Діріхле для рівняння Лапласа з новим лапласіаном. Побудовано загальну схему розв'язання задачі Діріхле в кулі у двовимірному та тривимірному просторах у випадку щільностей, які є інваріантними відносно ортогональної групи перетворень. Крім того, знайдено явні розв'язки задачі для досить багатих та важливих класів щільностей, серед яких є і гаусівська щільність.

Ключові слова: міра; дивергенція; лапласіан; рівняння Лапласа; задача Діріхле.